AN EXTENSION OF THE WRIGHT'S 3/2-THEOREM FOR THE KPP-FISHER DELAYED EQUATION

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ABSTRACT. We present a short proof of the following natural extension of the famous Wright's 3/2-stability theorem: the conditions $\tau \leq 3/2$, $c \geq 2$ imply the presence of the positive traveling fronts (not necessarily monotone) $u = \phi(x \cdot \nu + ct)$, $|\nu| = 1$, in the delayed KPP-Fisher equation $u_t(t,x) = \Delta u(t,x) + u(t,x)(1 - u(t-\tau,x))$, $u \geq 0$, $x \in \mathbb{R}^m$.

1. Introduction and main result

The delayed KPP-Fisher (i.e. Kolmogorov-Petrovskii-Piskunov-Fisher) equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)(1 - u(t-\tau,x)),\ u \geq 0,\ x \in \mathbb{R}^m, \tau \geq 0, \tag{1.1}$$

is one of the most conspicuous examples of delayed reaction-diffusion equations. During the past decade, this model together with the following non-local version of the KPP-Fisher equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)\left(1 - \int_{\mathbb{R}^n} K(y)u(t,x-y)dy\right),\ \int_{\mathbb{R}^n} K(s)ds = 1. \tag{1.2}$$

have been intensively studied by many authors, see e.g. [2, 3, 4, 5, 7, 9, 11, 17].

One of the key topics related to equations (1.1), (1.2) concerns the existence and further properties of smooth positive traveling front solutions $u(x,t) = \phi(\nu \cdot x + ct)$, $|\nu| = 1$ for (1.1). It is supposed that $c > 0$ and that the profile ϕ satisfies the boundary conditions $\phi(-\infty) = 0$, $\phi(+\infty) = 1$. A few years ago, not much was known about the conditions guaranteeing the existence of these wavefronts in (1.1). Several existence results having rather partial character were provided in [17] (for each $c > 2$ and $\tau \in [0, \tau(c)]$ with sufficiently small $\tau(c)$) and in [5, 6] (for each $\tau \leq 3/2$ and $c \geq c(\tau)$ with sufficiently large $c(\tau)$). In this respect, a significant progress was achieved only very recently when the existence and uniqueness problems for (1.1), (1.2) were completely solved for the case of monotone profiles [2, 3, 4, 7, 9, 11]. However, the monotonicity of ϕ is a rather restrictive assumption: it is clear that traveling fronts of (1.1), (1.2) that oscillate around 1 at $+\infty$ (hence, non-monotone ones) comprise the largest part of the set of all wavefront solutions [1, 2, 9, 15]. In this note, by establishing an 'almost optimal criterion' for the presence of oscillating fronts in equation (1.1), we achieve an essential improvement of the existence results from [6, 7, 11, 17]. Still, the complete solution of the mentioned problem remains to be a quite challenging project which is directly connected to the long standing Wright's global stability conjecture [10, 16].

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Let us explain the last comment in more detail. Indeed, looking for a wave solution of (1.1) in the slightly modified form $u(t,x) = \psi(\sqrt{\epsilon}x + t)$, $\epsilon = 1/c^2$, $\psi(s) = \phi(cs)$, we find that

$$\epsilon\psi''(t) - \psi'(t) + \psi(t)(1 - \psi(t-\tau)) = 0, \ t \in \mathbb{R}. \tag{1.3}$$

In the limit case $\epsilon = 0$ equation (1.3) is called the Hutchinson's equation and it was conjectured by E.M. Wright [16] that the steady state $\psi = 1$ of (1.3) with $\epsilon = 0$ is globally stable in the domain of all positive solutions $\psi > 0$ if and only if $\tau \leq \pi/2$. A weaker version of the Wright's conjecture can be also considered: the Hutchinson's equation has a positive heteroclinic connection (i.e. traveling front type solution) if and only if $\tau \leq \pi/2$. The both conjectures are supported by the *'very difficult theorem of Wright'* (the quoted phrase is from the Jack Hale's book [8, p.64]) proved in [16]: If $\tau \leq 3/2$ then the positive equilibrium of (1.3) with $\epsilon = 0$ is globally stable in the domain of positive solutions. Remarkably, as it was shown in [5, 6] by means of the Hale-Lin approach, the Wright's 3/2-theorem can be extended to (1.3) with $\epsilon > 0$ in the following way: equation (1.3) has a positive heteroclinic connection for each positive fixed $\tau \leq 3/2$ if $\epsilon > 0$ is sufficiently small. The main result of this work shows that the smallness condition on ϵ (i.e. the requirement that the propagation speed c has to be sufficiently large) can be avoided and that the full analog of the Wright's theorem holds for (1.1):

Theorem 1.1. *Assume that $c \geq 2$ and $\tau \in [0, 3/2]$. Then the delayed KPP-Fisher reaction-diffusion equation (1.1) has at least one positive traveling front solution.*

It is well known that the inequality $c \geq 2$ is mandatory for the existence of positive wavefronts [2, 7, 9]. We also believe that, similarly to the monotone fronts [3, 4, 7, 9], there is a unique (up to a translation) oscillating front for each fixed c.

Theorem 1.1 strongly supports the next generalisation of the weak Wright's conjecture [9]: equation (1.1) has at least one positive traveling front $u = \phi(\nu \cdot x + ct)$, $|\nu| = 1$, if and only if $c \geq 2$ and the equation $\lambda^2 - c\lambda - e^{-\lambda c\tau} = 0$ has a unique root λ with the positive real part. In particular, this means that the maximal possible improvement of the interval $[0, 3/2]$ in Theorem 1.1 is $[0, \pi/2]$, see [9, Figure 1].

The starting point for the proof of Theorem 1.1 is the fact that, for each $c \geq 2$, equation (1.1) (similarly to equation (1.2), see [2]) has at least one positive wave solution $u = \phi(\nu \cdot x + ct)$, $|\nu| = 1$, satisfying the boundary conditions $\phi(-\infty) = 0$, $0 < m = \liminf_{t\to+\infty} \phi(t) \leq \limsup_{t\to+\infty} \phi(t) = M < +\infty$ (i.e. a semi-wavefront), see [9]. The next important fact is that each non-monotone semi-wavefront profile is sine-like slowly oscillating around 1 at $+\infty$ [9, 13, 14]. In Section 3, we present several explicit analytic relations limiting the amplitude of these oscillations. At the first glance, the mentioned restrictions are generated by rather cumbersome bounding functions. Surprisingly, these functions have nice internal structures (previously analysed in [12]) that allow for their satisfactory description in Section 2. At the very end of Section 3, in order to demonstrate Theorem 1.1, we show that $\tau \leq 3/2$ together with $c \geq 2$ imply $m = M = 1$.

2. Auxiliary functions

Our approach to the proof of Theorem 1.1 requires the construction of several suitable bounding functions. These functions are necessary to relate the values of m and M (defined a few lines above); it is clear that their choice is by no means unique. Below, we present our auxiliary functions and prove their properties which are later used in the proof of Theorem 1.1. First, we consider

$$\rho(x) = \rho(x,\tau,c) := \tau c f(w(x)), \text{ where } f(x) := \frac{-c+\sqrt{c^2+4x}}{2}, \quad w(x) := e^{-x}-1.$$

Proposition 2.1. [9] Let $c \geq 2$. Then the real analytic function $\rho(x)$, $x \in \mathbb{R}$, $\rho(0) = 0$, $\rho(-\infty) = +\infty$, $\rho(+\infty) = -0.5\tau c(-c+\sqrt{c^2-4}) < 0$, is strictly decreasing (in fact, $\rho'(x) < 0$, $x \in \mathbb{R}$) and has the negative Schwarz derivative $(S\rho)(x)$ on $\mathbb{R}$: $(S\rho)(x) = \rho'''(x)/\rho'(x) - 3/2\,(\rho''(x)/\rho'(x))^2 < 0, \quad x \in \mathbb{R}$.

It is straightforward to see that ρ is a convex function:

$$\rho''(x) = c\tau e^{-x}(f'(w(x)))^3(c^2-4+2e^{-x}) > 0, \quad x \in \mathbb{R}.$$

Corollary 2.2. If $c \geq 2$ then, for all $x > 0$, it holds that

$$\rho(x) > r(x) := \frac{\rho'(0)x}{1-0.5\rho''(0)x/\rho'(0)} = \frac{-\tau x}{1+0.5(1-2/c^2)x}.$$

Proof. It is an immediate consequence of Proposition 2.1 and [12, Lemma 2.1]. $\square$

Next, for each $c \geq 2$, $\tau \in (1, 3/2]$, we consider

$$A_-(x,c,\tau) = x + \rho(x) + \frac{1}{\rho(x)}\int_x^0 \rho(s)ds, \quad x \neq 0;$$

$$A_+(x,c,\tau) = x + r(x) + \frac{1}{r(x)}\int_x^0 r(s)ds, \quad B(x,c,\tau) := \frac{1}{r(x)}\int_{-r(x)}^0 r(s)ds, \quad x > 0.$$

It is easy to see that $A_\pm, B$ are continuous at $x = 0$ if we set $A_\pm(0,c,\tau) = B(0,c,\tau) = 0$. Observe also that $B(x,c,\tau)$ is strictly decreasing on $\mathbb{R}_+$,

$$A'_\pm(0,c,\tau) = \frac{1}{2} - \tau, \quad A''_\pm(0,c,\tau) = (\tau - \frac{1}{6})(1 - \frac{2}{c^2}),$$

$$\begin{aligned} A_-(x,c,\tau) &< A_-(x,c,3/2), \ x < 0, \ \tau > 1; \\ A_+(x,c,\tau) &> A_+(x,c,3/2), \ x > 0, \ \tau > 1; \\ B(x,c,\tau) &> B(x,c,3/2), \ x > 0, \ \tau > 1. \end{aligned} \tag{2.1}$$

Let $x_2 > 0$ be the unique positive solution of equation $-r(x) = x$. Since $\tau > 1$, it holds, for a positive x, that $x/r(x) > -1$ if and only if $x \in (0, x_2)$. As it was established in [12, Lemma 2.3], $A_+(x,c,\tau)$ is strictly decreasing in the first variable on $(-\infty, x_2]$. The next result has a similar proof:

Lemma 2.3. *$A'_-(x,c,\tau) < 0$ and $(SA_-)(x,c,\tau) < 0$ once $x/\rho(x) > -1$.*

Proof. Using the convexity of ρ and recalling that $-\rho'(0) = \tau > 1$, $\rho(0) = 0$, it is easy to see that $x/\rho(x) > -1$ if and only if $x < \bar{x}_2$ where $\bar{x}_2$ is the unique positive solution of equation $-\rho(x) = x$. In consequence, $x\rho(x) + \rho^2(x) > 0$, $x < \bar{x}_2$, $x \neq 0$,

$$A'_-(x,c,\tau) = \rho'(x)\left(1 - \frac{\int_x^0 \rho(s)ds}{\rho^2(x)}\right) < \rho'(x)\left(1 + \frac{x\rho(x)}{\rho^2(x)}\right) < 0, \ x < \bar{x}_2, \ x \neq 0.$$

We know also $A'_-(0,c,\tau) = 0.5 - \tau < 0$. Now, integrating by parts, we obtain

$$A_-(x,c,\tau) = \rho(x) + \frac{x\rho(x) + \int_{\rho(x)}^0 v d\theta(v)}{\rho(x)} = \rho(x) + \frac{1}{\rho(x)}\int_0^{\rho(x)} \theta(v)dv = G(\rho(x)),$$

where $\theta(v) := \rho^{-1}(v)$ and $G(z) = z + \int_0^1 \theta(vz)dv$.

Then, by Proposition 2.1 and the formula for the Schwarzian derivative of the composition of two functions, we obtain

$$(SA_-)(x,c,\tau) = (SG)(\rho(x))(\rho'(x))^2 + (S\rho)(x) < (SG)(\rho(x))(\rho'(x))^2.$$

Thus the negativity of SA_- will follow from the inequality $(SG)(\rho(x)) < 0$. Since $A'_-(x,c,\tau) < 0$ if and only if $G'(\rho(x)) > 0$, it suffices to show that $(SG)(\rho(x)) < 0$ when $G'(\rho(x)) > 0$. Now, in view of Proposition 2.1,

$$\theta'''(\rho(x)) = \frac{3(\rho''(x))^2 - \rho'''(x)\rho'(x)}{(\rho'(x))^5} = \frac{-(S\rho)(x)}{(\rho'(x))^3} + \frac{3}{2}\frac{(\rho''(x))^2}{(\rho'(x))^5} < 0.$$

Hence, $G'''(z) = \int_0^1 v^3\theta'''(vz)dv < 0,\ z = \rho(x)$, and therefore $(SG)(\rho(x)) < G'''(\rho(x))/G'(\rho(x)) < 0$. This completes the proof of Lemma 2.3. □

Next, for $c \geq 2,\ \tau \in (1, 3/2]$, we will also consider the functions

$$R(x,c,\tau) = \frac{A'_+(0,c,\tau)x}{1 - 0.5A''_+(0,c,\tau)x/A'_+(0,c,\tau)},$$

$$D(x,c,\tau) = \begin{cases} A_-(x,c,\tau) & \text{if } x \leq 0, \\ A_+(x,c,\tau) & \text{if } x \in [0, x_2], \\ B(x,c,\tau) & \text{if } x \geq x_2. \end{cases}$$

As the above discussion shows, $D(x,c,\tau)$ is strictly decreasing in $x \in \mathbb{R}$. From now on, we fix $\tau = 3/2$ and set $A_\pm(x,c) := A_\pm(x,c,3/2)$, $B(x,c) := B(x,c,3/2)$, $D(x,c) := D(x,c,3/2)$, $R(x,c) := R(x,c,3/2)$. The strictly decreasing function $D(x,c)$ has the following additional nice property:

Proposition 2.4. If $c \geq 2$ then $D(x,c) > R(x,c)$ for all $x > 0$.

Proof. The above inequality follows from [12, Corollary 2.7] if we take there $\mathfrak{f}'(0) = r'(0) = -\tau = -3/2$. It should be observed that the definitions of functions A, B, r, R in [12] are identical to the definitions of A_+, B, r, R in this paper. The only formal difference with [12] is the presence of parameter c in the expressions for the second derivatives of A_+, r, R at 0. However, once these derivatives are positive, the proofs in [12] do not matter on their exact values , e.g. see Lemma 2.6 from [12]. □

Corollary 2.5. $F(x) := A_-(R(x)) < x$ for all $x > 0$.

Proof. By Lemma 2.3, $(SF)(x) = (SA_-)(R(x))(R'(x))^2 < 0$ for all x from some open neighbourhood of $[0, +\infty)$. Also $SR \equiv 0$, so that

$$0 = (SR)(0) = -R'''(0) - 1.5(R''(0))^2 = -R'''(0) - 1.5(A''_-(0))^2.$$

Next, we have that $F'(0) = -R'(0) = 1,\ F''(0) = A''_-(0)(R'(0))^2 + A'_-(0)R''(0) = 0$,

$$F'''(0) = A'''_-(0)(R'(0))^3 + 3A''_-(0)R'(0)R''(0) + A'_-(0)R'''(0) = (SA_-)(0) < 0.$$

Therefore $F(x) < x$ for all small positive x. Now, suppose that $F(z) = z$ for some leftmost positive z. Then $F'(z) \geq 1$ and therefore function $y = F'(x) > 0$, $x \in [0, z]$, $F'(0) = 1$, has a positive local minimum at some point $p \in (0, d)$. But then $F''(p) = 0,\ F'''(p) \geq 0$, and in this way $(SF)(p) \geq 0$, a contradiction. □

3. Bounding relations and the convergence of semi-wavefronts

As we have mentioned in the introduction, for each $c \geq 2$, equation (1.1) has at least one positive wave $u = \phi(\nu \cdot x + ct)$, $|\nu| = 1$, satisfying the boundary conditions $\phi(-\infty) = 0$, $0 < \liminf_{t\to+\infty} \phi(t) \leq \limsup_{t\to+\infty} \phi(t) < +\infty$. Clearly, ϕ satisfies

$$\phi''(t) - c\phi'(t) + \phi(t)(1 - \phi(t-h)) = 0, \ h := c\tau, \ t \in \mathbb{R}. \tag{3.1}$$

The change of variables $\phi(t) = e^{-x(t)}$ transforms the latter equation into

$$x''(t) - cx'(t) - (x'(t))^2 + (e^{-x(t-h)} - 1) = 0, \ t \in \mathbb{R}. \tag{3.2}$$

By Theorem 4 from [9], $x(t)$ is sine-like oscillating around 0. More precisely, there exists an increasing sequence $Q_j, \ j \geq 0$, of zeros of $x(t)$ such that $x(t) < 0$ on $(Q_0, Q_1) \cup (Q_2, Q_3) \cup \dots$ and $x(t) > 0$ on $(-\infty, Q_0) \cup (Q_1, Q_2) \cup (Q_3, Q_4) \cup \dots$ Furthermore, $x(t)$ has exactly one critical point (hence, local extremum point) T_j on each interval $[Q_j, Q_{j+1}]$ and $T_j - Q_j < h$ for all j. Hence, $y(t) := x'(t)$ does not change its sign on the intervals $(T_j, T_{j+1}), \ j = 0, 1, 2 \dots$ and $y(T_j) = 0$. Therefore y solves the boundary value problem

$$y' = y^2 + cy - g(t), \quad y(T_j) = y(T_{j+1}) = 0, \tag{3.3}$$

where $c \geq 2$ and $g(t) := w(x(t-h))$ is C^2-smooth on $\mathbb{R}$.

Lemma 3.1. *For each integer $j \geq 0$, solution $y(t)$ has a unique critical point (absolute minimum point) $p_j \in [T_{2j+1}, T_{2j+2}]$, and, for all $t \in (p_j, T_{2j+2})$, it holds $y(t) > \rho(x(t-h))/h$. Furthermore, for each non-increasing function $M = M(t)$, $t \in [Q_{2j}, Q_{2j+2}]$, such that $x(t) \leq M(t), \ t \in [Q_{2j}, Q_{2j+2}]$, it holds*

$$y(t) > \rho(M(t-h))/h, \ t \in (T_{2j+1}, T_{2j+2}) \setminus \{p_j\}. \tag{3.4}$$

Similarly, for $j \geq 1$, solution $y(t)$ has a unique critical point (absolute maximum point) $q_j \in [T_{2j}, T_{2j+1}]$, and, for all $t \in (q_j, T_{2j+1})$, it holds $y(t) < \rho(x(t-h))/h$. Furthermore, for each non-decreasing function $m = m(t), t \in [Q_{2j-1}, Q_{2j+1}]$, such that $x(t) \geq m(t), \ t \in [Q_{2j-1}, Q_{2j+1}]$, it holds

$$y(t) < \rho(m(t-h))/h, \ t \in (T_{2j}, T_{2j+1}) \setminus \{q_j\}.$$

Proof. We will prove only the first assertion of the lemma, the proof of the second statement being completely analogous. So let us consider the slope field for differential equation (3.3). Two zero isoclines

$$\lambda_1(t) = \frac{-c - \sqrt{c^2 + 4g(t)}}{2} < -\frac{c}{2} < \frac{-c + \sqrt{c^2 + 4g(t)}}{2} := \lambda_2(t)$$

partition the plane $\mathbb{R}^2$ into three horizontal bands

$$\Pi_1 = \{(t, y) : y \leq \lambda_1(t)\}, \Pi_2 = \{(t, y) : \lambda_1(t) \leq y \leq \lambda_2(t)\}, \Pi_3 = \{(t, y) : y \geq \lambda_2(t)\},$$

limited by the graphs of functions $y = \lambda_1(t), y = \lambda_2(t)$. We observe that the portions of integral curves of (3.3) belonging to the interior of domains Π_1, Π_3 [respectively, Π_2] are increasing [respectively, decreasing]. Since $y(T_{2j+2}) = 0$ and $g(T_{2j+2}) = \exp(-x(T_{2j+2} - h)) - 1 < 0$ we find that $(T_{2j+2}, 0) \in \operatorname{Int}\Pi_3$, where $\operatorname{Int} X$ denotes the interior part of the set X. Similarly, $(T_{2j+1}, 0) \in \operatorname{Int}\Pi_2$ while the points T_{2j+1} and T_{2j+2} are separated by a unique zero $Q_{2j+1} + h$ of $y = \lambda_2(t)$ on $[T_{2j+1}, T_{2j+2}]$. As a consequence, the integral curve of each function $y(t)$ solving (3.3) never enters Π_1 and belongs to $\Pi_2 \cup \Pi_3$. Moreover, it is clear that $y'(t) > 0$ on some maximal interval (p_j, T_{2j+2}) where $y(p_j) = \lambda_2(p_j), \ y'(p_j) = 0$. Since

clearly $0 \leq y''(p_j) = -g'(p_j)$, the point $(p_j, \lambda_2(p_j))$ lies on the decreasing part of the graph Γ of $y = \lambda_2(t)$ (observe that $\lambda_2'(t) = g'(t)/\sqrt{c^2+4g(t)}$). We claim that $(t, y(t))$ does not cross Γ again for all $t \in [T_{2j+1}, p_j)$. Indeed, otherwise there exists some $d \in [Q_{2j+1}+h, p_j)$ such that $y(d) = \lambda_2(d)$ and therefore $y'(d) = 0$ while $\lambda'(d_2) = g'(d)/\sqrt{c^2+4g(d)} < 0$ since $g'(t) = -x'(t-h)\exp(-x(t-h)) < 0$, $t \in [Q_{2j+1}+h, p_j),\ g(p_j) \leq 0$. This means that at the moment $t = d$ the integral curve of the solution $y = y(t)$ intersects transversally Γ, enters the domain Π_3 and is strictly increasing on (d, p_j). Since $y = \lambda_2(t)$ is strictly decreasing on the same interval, we get a contradiction: $y(p_j) > \lambda_2(p_j)$.

Hence, we have the following description of the behaviour of each solution $y(t)$ to (3.3) on $[T_{2j+1}, T_{2j+2}]$: there exists a point $p_j \in (T_{2j+1}, T_{2j+2})$ such that

i) $y'(t) > 0,\ y(t) > \lambda_2(t) = \rho(x(t-h))/h, \quad t \in (p_j, T_{2j+2}]$;

ii) $y'(p_j) = 0,\ y(p_j) = \lambda_2(p_j)$;

iii) $y'(t) < 0,\ y(t) < \lambda_2(t), \quad t \in [T_{2j+1}, p_j)$.

Finally, in order to justify (3.4), we observe that $[Q_{2j}, Q_{2j+2}] \supset [T_{2j+1}-h, T_{2j+2}-h]$. Therefore, since ρ decreases on $\mathbb{R}$, we obtain that $\rho(x(t-h)) \geq \rho(M(t-h))$ for $t \in [T_{2j+1}, T_{2j+2}]$. Thus the property i) implies (3.4) for all $t \in (p_j, T_{2j+2}]$. In particular, $y(p_j) \geq \rho(M(p_j - h))/h$. Since, in addition, $y(t)$ is strictly decreasing on $[T_{2j+1}, p_j)$, $\rho(M(t-h))$ is non-decreasing on the same interval, we conclude that (3.4) also holds for all $t \in [T_{2j+1}, p_j)$. This completes the proof of Lemma 3.1. □

Remark 3.2. For the oscillating semi-wavefront solutions $x = x(t)$ of equation (3.2), the above result improves considerably the estimations of Lemma 20 from [9]. In order to obtain such an improvement, here we have used our knowledge of slowly oscillating behaviour of $g(t)$: this information was not relevant for the proof of Lemma 20.

Corollary 3.3. The profiles of oscillating semi-wavefronts to equation (1.1) have a unique inflection point between each two consecutive extremum points.

In the next stage of our studies, we will evaluate the extremal values $V_j = x(T_j)$ for $j \geq 1$ (it follows from [9, Corollary 16] that $V_0 \geq -ch$).

Lemma 3.4. *Let $c \geq 2,\ \tau \in (1, 3/2]$ and $x(t) = -\ln\phi(t)$ oscillates on $[Q_0, +\infty)$. Then $V_{2j+1} \leq A_-(V_{2j}, c, \tau),\ j \geq 0$, $V_{2j} \geq B(V_{2j-1}, c, \tau),\ j \geq 1$. If, in addition, $V_{2j-1} \leq x_2$, then $V_{2j} \geq A_+(V_{2j-1}, c, \tau),\ j \geq 1$.*

Proof. As we know, $V_1 = x(T_1) > 0$ with $T_1 - Q_0 > h$ and $x'(T_1) = 0,\ x(Q_1) = 0$, $T_1 - Q_1 < h$. Set $Q_{-1} = T_{-1} = -\infty$, it is clear that $x(s) \geq V_0$ for all $s \in [Q_{-1}, Q_1]$. On the other hand, due to Lemma 3.1, we know that

$$x'(t) \leq \max_{s\in[T_0,T_1]} x'(s) \leq \frac{1}{h}\rho(\min_{s\in[Q_{-1},Q_1]} x(s)) \leq \frac{1}{h}\rho(V_0), \quad t \in [T_0, T_1],$$

and therefore

$$x(t) = -\int_t^{Q_1} x'(s)ds \geq -\frac{1}{h}\int_t^{Q_1} \rho(V_0)ds = \frac{\rho(V_0)}{h}(t - Q_1) = \tilde{m}(t), \quad t \in [T_0, Q_1].$$

In particular, $x(T_0) = V_0 \geq \tilde{m}(T_0)$ and therefore equation $\tilde{m}(t) = V_0$ has a root $t_1 \in [T_0, Q_1]$. Since $V_0 < 0$, we know from the first lines of the proof of Lemma 2.3 that $t_1 - Q_1 = hV_0/\rho(V_0) > -h$. Consider now the non-decreasing function

$$m(t) = \begin{cases} \tilde{m}(t) & \text{if } t \in [t_1, Q_1] \subset (Q_1 - h, Q_1], \\ V_0 & \text{if } t \leq t_1, \end{cases}$$

it is clear that $x(t) \geq m(t)$ for all $t \in [Q_{-1}, Q_1]$. Therefore, by Lemma 3.1,

$$V_1 = \int_{Q_1}^{T_1} x'(s)ds \leq \frac{1}{h}\int_{Q_1}^{T_1} \rho(m(s-h))ds \leq \frac{1}{h}\int_{Q_1-h}^{Q_1} \rho(m(s))ds = A_-(V_0, c, \tau).$$

Next, consider $V_2 = x(T_2) < 0$, we have $x'(t) < 0$ on (T_1, T_2), $x'(T_2) = 0$, $x(Q_2) = 0$ and $T_2 - Q_2 < h$. By Lemma 3.1 and Corollary 2.2,

$$x'(t) \geq \min_{s\in[T_1,T_2]} x'(s) \geq \frac{1}{h}\rho(\max_{s\in[Q_0,Q_2]} x(s)) \geq \frac{\rho(V_1)}{h} > \frac{r(V_1)}{h}, \quad t \in [T_1, T_2],$$

and therefore

$$x(t) = -\int_t^{Q_2} x'(s)ds < \frac{r(V_1)}{h}(t - Q_2) = \tilde{M}(t), \quad t \in [T_1, Q_2).$$

Since $\tilde{M}(t)$ is decreasing on $[Q_0, Q_2]$, it holds that $\tilde{M}(T_1) > x(T_1) = V_1$ and $\max_{t\in[Q_0,Q_2]} x(t) = x(T_1)$, we have that $x(t) < \tilde{M}(t)$ for $t \in (Q_0, Q_2]$. Then Lemma 3.1 yields

$$V_2 = \int_{Q_2}^{T_2} x'(s)ds \geq \frac{1}{h}\int_{Q_2-h}^{Q_2} \rho(\tilde{M}(s))ds > \frac{1}{h}\int_{Q_2-h}^{Q_2} r(\tilde{M}(s))ds = B(V_1, c).$$

Suppose now that $V_1 \leq x_2$. Let $t = t_2$ solve the equation $hV_1 = r(V_1)(t - Q_2)$, then $t_2 - Q_2 = hV_1/r(V_1) \geq -h$ (see the comments following the definition of x_2). Consider the non-increasing function

$$M(t) = \begin{cases} \tilde{M}(t) & \text{if } t \in [t_2, Q_2] \subset [Q_2 - h, Q_2], \\ V_1 & \text{if } t \leq t_2, \end{cases}$$

it is clear that $x(t) \leq M(t)$ for all $t \in [Q_0, Q_2]$. By applying Lemma 3.1 and Corollary 2.2, we obtain

$$V_2 = \int_{Q_2}^{T_2} x'(s)ds \ > \frac{1}{h}\int_{Q_2-h}^{Q_2} \rho(M(s))ds > \frac{1}{h}\int_{Q_2-h}^{Q_2} r(M(s))ds = A_+(V_1, c).$$

Finally, we can repeat the above arguments to obtain similar estimations for all $j > 2$. This completes the proof of Lemma 3.4. □

We are now in a position to finalise the proof of Theorem 1.1. Consider the following finite limits

$$0 \geq m_* = \liminf_{j\to+\infty} V_j = \liminf_{t\to+\infty} x(t), \quad 0 \leq M_* = \limsup_{j\to+\infty} V_j = \limsup_{t\to+\infty} x(t).$$

From Lemmas 2.3 and 3.4 we deduce that $M_* \leq A_-(m_*, c, \tau)$ and $m_* \geq D(M_*, c, \tau)$. Clearly, Theorem 1.1 will be proved if we show that $\tau \leq 1.5$ yields $M_* = 0$. Since this implication was already proved for $\tau \leq 1$ in [9, Theorem 8], we may assume that $\tau > 1$. So let us suppose that $M_* > 0$, $\tau \in (1, 3/2]$. But then, due to inequalities (2.1) and Proposition 2.4, $M_* \leq A_-(m_*, c, \tau) < A_-(m_*, c)$ and $m_* \geq D(M_*, c, \tau) > D(M_*, c) > R(M_*)$. Therefore we have $M_* < A_-(R(M_*))$. However, by Corollary 2.5, $A_-(R(M_*)) < M_*$, a contradiction. Hence, $M_* = 0$ and the proof of Theorem 1.1 is completed.

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